Probability & Statistics (1)

Jointly Distributed Random Variables (I)

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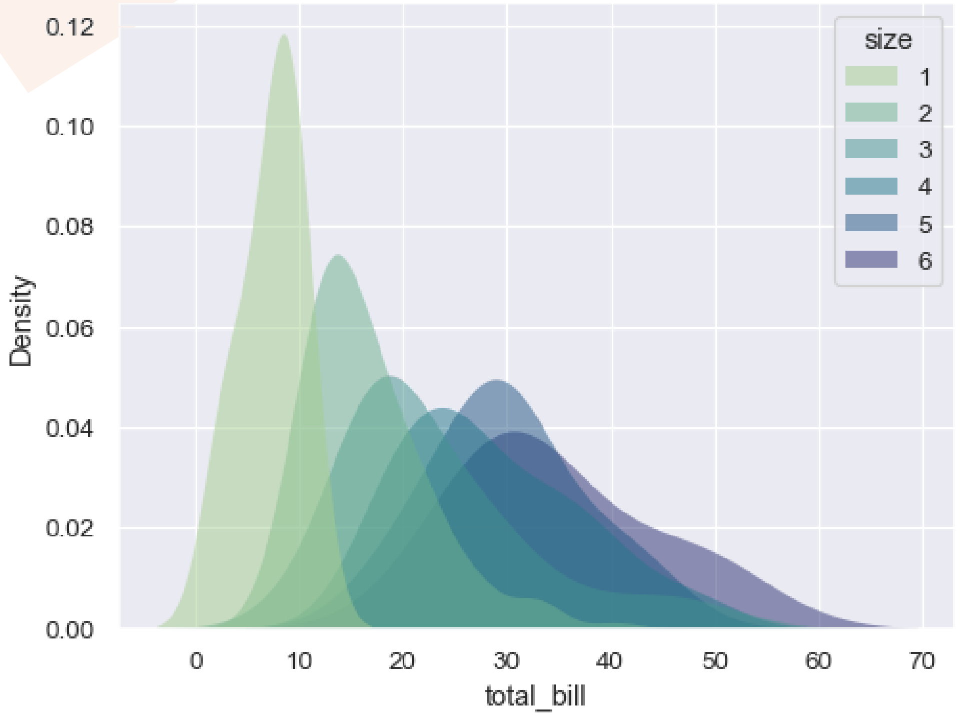
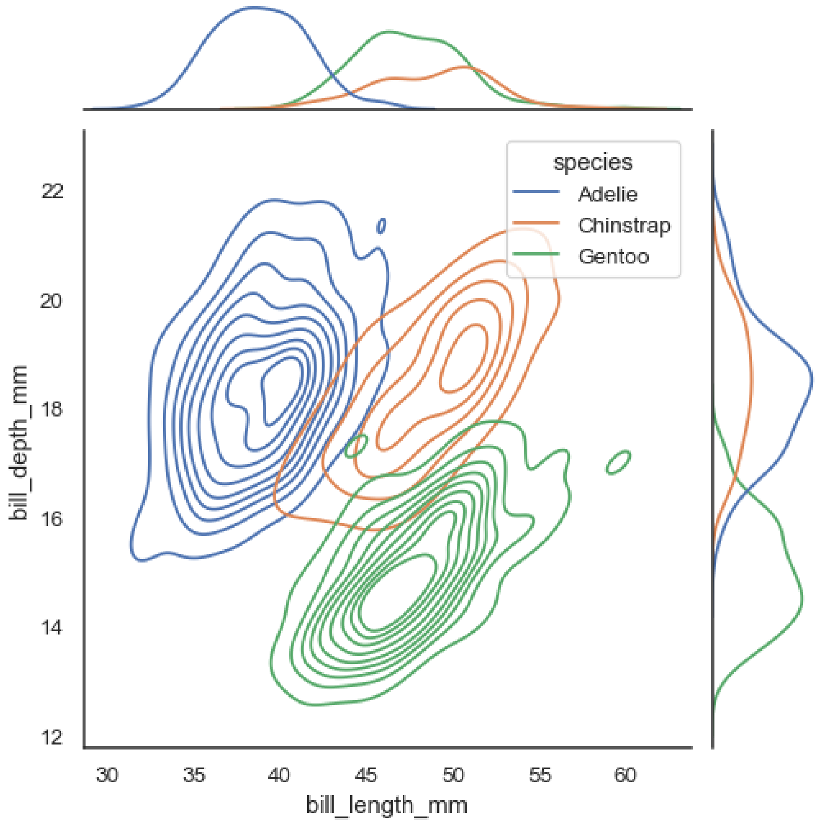
Joint Distribution Functions

- 之前我們都只有考慮到一個隨機變數，如果今天我們需要同時考慮兩個隨機變數結合起來的機率分布，這個時候我們就會使用到Joint Distribution Functions。





Joint Distribution Functions

- 假設我們今天給定兩個隨機變數： X與Y，那麼他們的joint cumulative probability distribution function可以被定義為:

$$F(a,b) = P\{X \le a, Y \le b\}, where -\infty < a, b < \infty$$

- 如果我們想要求X的distribution的話:

$$F_X(a) = P\{X \le a\} = P\{X \le a, Y \le \infty\} = P\left(\lim_{b\to\infty}\{X \le a, Y \le \infty\}\right)$$

$$= \lim_{b\to\infty} P\{X \le a, Y \le \infty\} = \lim_{b\to\infty} F(a,b) = F(a,\infty)$$

- 同理可證，如果我們要求Y的distribution的話:

$$F_Y(b) = P\{Y \le b\} = \lim_{a\to\infty} F(a,b) = F(\infty,b)$$

Joint Distribution Functions

- 通常我們會稱F_X與F_Y的distribution functions為X與Y的marginal distributions。

$$
\begin{aligned}
&P\{X > a, Y > b\} = 1 - P(\{X > a, Y > b\}^c) \\
&= 1 - P(\{X > a\}^c \cup \{Y > b\}^c) \\
&= 1 - P(\{X \le a\} \cup \{Y \le b\}) \\
&= 1 - [P\{X \le a\} + P\{Y \le b\} - P\{X \le a, Y \le b\}] \\
&= 1 - F_X(a) - F_Y(b) + F(a, b)
\end{aligned}
$$

Joint Distribution Functions

- 如果今天我們的隨機變數X與Y都是離散的時候，則X與Y 的joint probability mass function為

$$p(x,y) = P\{X = x, Y = y\}$$

- X的probability mass function可以從$p(x,y)$得出

$$p_X(x) = P\{X = x\} = \sum_{y:p(x,y)>0} p(x,y)$$

- 所以Y的probability mass function為:

$$p_Y(y) = P\{Y = y\} = \sum_{x:p(x,y)>0} p(x,y)$$

Joint Distribution Functions

- **範例一**

假設隨機從摸彩桶取出3張摸彩券，而這個桶子中有3張一獎，4張二獎與5張銘謝惠顧。我們令隨機變數X與Y分別為抽到一獎與二獎的張數，X與Y的joint PMF為: $p(i,j) = P\{X = i, Y = j\}$，試求其joint PMF的數值。

Solution:

$$p(0,0) = \frac{\binom{5}{3}}{\binom{12}{3}} = \frac{10}{220}; p(0,1) = \frac{\binom{4}{1}\binom{5}{2}}{\binom{12}{3}} = \frac{40}{220}; p(0,2) = \frac{\binom{4}{2}\binom{5}{1}}{\binom{12}{3}} = \frac{30}{220}; p(0,3) = \frac{\binom{4}{3}}{\binom{12}{3}} = \frac{4}{220}$$

$$p(1,0) = \frac{\binom{3}{1}\binom{5}{2}}{\binom{12}{3}} = \frac{30}{220}; p(2,0) = \frac{\binom{3}{2}\binom{5}{1}}{\binom{12}{3}} = \frac{15}{220}; p(3,0) = \frac{\binom{3}{3}}{\binom{12}{3}} = \frac{1}{220};$$

Joint Distribution Functions

$$p(1,1)=\frac{\binom{3}{1}\binom{4}{1}\binom{5}{1}}{\binom{12}{3}}=\frac{30}{220};p(1,2)=\frac{\binom{3}{1}\binom{4}{2}}{\binom{12}{3}}=\frac{18}{220};p(2,1)=\frac{\binom{3}{2}\binom{4}{1}}{\binom{12}{3}}=\frac{12}{220};$$

Table 1 $P\{X=i,Y=j\}$

i \ j	**0**	**1**	**2**	**3**	**Row sum**=$P\{X=i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column sum=$P\{Y=j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	1

Joint Distribution Functions

- **範例二**

假設今天做人口抽樣調查，15%家庭中沒有小孩，20%有一個小孩，35%有兩個小孩，30%有三個小孩。假設小孩是男生或是女生的機率相等，那麼隨機抽樣一個家庭，B為小孩為男生的數量；G為小孩為女生的數量，則$P\{B=i,G=j\}$的機率為何?

Solution:

$$P\{B=0,G=0\}=P\{no\ children\}=0.15$$

Joint Distribution Functions

Table $P\{B = i, G = j\}$

i \ j	0	1	2	3	Row Sum $= P\{B = j\}$
0					0.3750
1					0.3875
2					0.2000
3					0.0375
Column Sum $= P\{G = j\}$	0.3750	0.3875	0.2000	0.0375	

Joint Distribution Functions

- We say that X and Y are jointly continuous if there exists a function $f(x,y)$, defined for all real x and y, having the property that, for every set C of pairs of real numbers (that is, C is a set in the two-dimensional plane),

$$P\{(X,Y) \in C\} = \iint_{(x+y)\in C} f(x,y)dxdy$$

- The function $f(x,y)$ is called the joint probability density function of X and Y. If A and B are any set of real numbers, then, by defining $C = \{(x,y): x \in A, y \in B\}$

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x,y)dxdy$$

Joint Distribution Functions

$$F(a,b) = P\{X \in (-\infty, a], Y \in (-\infty, b]\} = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)dxdy$$

Differentiation

$$f(a,b) = \frac{\partial^2}{\partial a \partial b} F(a,b)$$

$$P\{a < X < a + da, b < Y < b + db\} = \int_{-\infty}^{b+db}\int_{-\infty}^{a+da} f(x,y)dxdy$$

$$\approx f(a,b)dadb$$

Joint Distribution Functions

$$P\{X \in A\} = P\{X \in A, Y \in (-\infty, \infty)\} = \int_A \int_{-\infty}^{\infty} f(x, y)dydx = \int_A f_X(x)dx$$

where

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy\,; \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

Joint Distribution Functions

- **範例三**

給定X與Y的joint density function為

$$f(x,y)=\begin{cases}2e^{-x}e^{-2y}, 0<x<\infty, 0<y<\infty\\ 0, otherwise\end{cases}$$

試問 $(a)\ P\{X>1,Y<1\}$; $(b)\ P\{X<Y\}$; $(c)\ P\{X<a\}$

Solution:

$$P\{X>1,Y<1\}=\int_0^1\int_1^\infty 2e^{-x}e^{-2y}\,dxdy=\int_0^1 2e^{-2y}\left(-e^{-x}\Big|_1^\infty\right)\mathrm{dy}$$

$$=\mathrm{e}^{-1}\int_0^1 2e^{-2y}\,dy=e^{-1}(1-e^{-2})$$

Joint Distribution Functions

(b)

$$P\{X < Y\} = \iint_{(x,y):x<y} 2e^{-x}e^{-2y}\,dxdy = \int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dxdy$$

$$= \int_0^\infty 2e^{-y}(1 - e^{-y})dy = \int_0^\infty 2e^{-y}dy - \int_0^\infty 2e^{-3y}dy = 1 - \frac{2}{3} = \frac{1}{3}$$

(c)

$$P\{X < a\} = \int_0^a \int_0^\infty 2e^{-x}e^{-2y}\,dydx = \int_0^a e^{-x}dx = 1 - e^{-a}$$

Joint Distribution Functions

- **範例四**

X 與 Y的joint probability定義如下:

$$f(x,y) = \begin{cases} e^{-(x+y)}, 0 < x < \infty, 0 < y < \infty \\ 0, otherwise \end{cases}$$

試問隨機變數X/Y的PDF為何?

Joint Distribution Functions

Solution:

令$a > 0$

$$F_{\frac{X}{Y}}(a) = P\left\{\frac{X}{Y} \le a\right\} = \iint_{\frac{x}{y}\le a} e^{-(x+y)}dxdy = \int_0^\infty \int_0^{ay} e^{-(x+y)}dxdy$$

$$= \int_0^\infty (1 - e^{-ay})e^{-y}dy = \{-e^{-y} + \frac{e^{-(a+1)y}}{a+1}\}$$

Differentiation shows that the density function of X/Y is given by

$$f_{\frac{X}{Y}}(a) = \frac{1}{(a+1)^2}, where\ 0 < a < \infty$$

Joint Distribution Functions

We can also define joint probability distributions for n random variables in exactly the same manner as we did for $n = 2$. For instance, the joint cumulative probability distribution function $F(a_1, a_2, \ldots, a_n)$ of the n random variables $X_1, X_2, \ldots, X_n$ is defined by

$$F(a_1, a_2, \ldots, a_n) = P\{X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n\}$$

Further, the n random variables are said to be jointly continuous if there exists a function $f(x_1, x_2, \ldots, x_n)$, called the joint probability density function, such that, for any set C in n-space,

$$P\{(X_1, X_2, \ldots, X_n) \in C\} = \iint_{(x_1, x_2, \ldots, x_n) \in C} \cdots \int f(x_1, \ldots x_n) dx_1 dx_2 \ldots dx_n$$

Joint Distribution Functions

In particular, for any n sets of real numbers A_1, A_2, …, A_n

$$P\{X_1 \in A_1, X_2 \in A_2, \dots, X_n \in A_n\}$$

$$= \int_{A_n} \int_{A_{n-1}} \dots \int_{A_1} f(x_1, \dots, x_n) dx_1 dx_2 \dots dx_n$$

Independent Random Variables

隨機變數X與Y彼此為獨立(independent)，則任何兩個集合的實數A與B

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}$$

令$E_A = \{X \in A\}, E_B = \{Y \in B\}$，因此對於所有的$a, b$

$$P\{X \leq a, Y \leq b\} = P\{X \leq a\}P\{Y \leq b\}$$

因此，X與Y的joint distribution function F，且X與Y相互獨立

$$F(a, b) = F_X(a)F_Y(b), for\ all\ a, b$$

當今天為離散隨機變數的時候

$$p(x, y) = p_X(x)p_Y(y), for\ all\ a, b$$

Independent Random Variables

$$P\{X \in A, Y \in B\} = \sum_{y \in B} \sum_{x \in A} p(x, y) = \sum_{y \in B} \sum_{x \in A} p_X(x)\, p_Y(y)$$

$$= \sum_{y \in B} p_Y(y) \sum_{x \in A} p_X(x) = P\{Y \in B\} P\{X \in A\}$$

$$f(x, y) = f_X(x) f_Y(y), for\ all\ x, y$$

Independent Random Variables

- **範例五**

假設有$n+m$次獨立試驗，成功的機會為p。X為在前面n次試驗中成功的次數，Y為在後面m次試驗中成功的次數。X與Y相互獨立。

$$P\{X=x,Y=y\}=\binom{n}{x}p^x(1-p)^{n-x}\binom{m}{y}p^y(1-p)^{m-y}$$

$$=P\{X=x\}P\{Y=y\}$$

相對地，X與Y為相互獨立，其中Z為在$n+m$有多少次成功。

Independent Random Variables

- **範例六**

給一段的時間區間中，進去郵局的男女性別機率為p與$1-p$為一個 Poisson random variable，其參數為λp與$\lambda(1-p)$。

Solution:

令隨機變數X與Y代表走進郵局的男女性分別數量。

$$P\{X=i,Y=j\}$$
$$=P\{X=i,Y=j|X+Y=i+j\}P\{X+Y=i+j\}$$
$$+P\{X=i,Y=j|X+Y\neq i+j\}P\{X+Y\neq i+j\}$$

According to …

$$P(E)=P(E|F)P(F)+P(E|F^c)P(F^c)$$

Independent Random Variables

Since $P\{X = i, Y = j | X + Y \neq i + j\} = 0$

$$P\{X = i, Y = j\} = P\{X = i, Y = j | X + Y = i + j\} P\{X + Y = i + j\}$$

Then,

$$P\{X + Y = i + j\} = e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!}$$

Given that $i + j$ people do enter the post office, since each person entering will be male with probability p, it follows that the probability that exactly i of them will be male.

$$P\{X = i, Y = j | X + Y = i + j\} = \binom{i+j}{i} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!}$$

Independent Random Variables

$$P\{X=i, Y=j\} = \binom{i+j}{i} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!}$$

$$= e^{-\lambda} \frac{(\lambda p)^i}{i!\,j!} [\lambda(1-p)]^j$$

$$P\{X=i\} = e^{-\lambda p} \frac{(\lambda p)^i}{i!} e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^j}{j!} = e^{-\lambda p} \frac{(\lambda p)^i}{i!}$$

Similarly,

$$P\{Y=j\} = e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^j}{j!}$$

Independent Random Variables

- **範例七**

一對情侶約去看世足，假設他們到達時間的時間為time uniformly distribution從中午12點到下午1點，試問兩個到達時間差距超過十分鐘的機率:

Solution:

$$2P\{X+10<Y\}=2\iint_{x+10<y} f(x,y)dxdy=2\iint_{x+10<y} f_X(x)f_Y(y)dxdy$$

$$=2\int_{10}^{60}\int_{0}^{y-10}\left(\frac{1}{60}\right)^2 dxdy=\frac{2}{60^2}\int_{10}^{60}(y-10)dy=\frac{25}{36}$$

Independent Random Variables

- **Proposition 1**

The continuous (discrete) random variables X and Y are independent if and only if their joint probability density function can be expressed.

$$f(x, y) = h(x)g(x), where \ -\infty < x < \infty, -\infty < y < \infty$$

Proof:

Let us give the proof in the continuous case. First, note that independence implies that the joint density is the product of the marginal densities of X and Y, so the preceding factorization will hold when the random variables are independent.

Independent Random Variables

Now, suppose that

$$f_{X,Y}(x,y) = h(x)g(y)$$

Then,

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dxdy = \int_{-\infty}^{\infty} h(x)\,dx \int_{-\infty}^{\infty} g(y)\,dy = C_1C_2$$

where $C_1 = \int_{-\infty}^{\infty} h(x)dx$ and $C_2 = \int_{-\infty}^{\infty} g(y)\,dy$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = C_2h(x)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = C_1g(y)$$

Since $C_1C_2 = 1$, it follows that $f(x,y) = f_X(x)f_Y(y)$

Independent Random Variables

- 範例八

If the joint density function of X and Y is

$$f(x, y) = 6e^{-2x}e^{-3y}, where\ 0 < x < \infty, 0 < y < \infty$$

And is equal to 0 outside this region, are the random variables independent? What if the joint density function is

$$f(x, y) = 24xy, where\ 0 < x < 1, 0 < y < 1, 0 < x + y < 1$$

and is equal to 0 otherwise.

Independent Random Variables

Solution:

In the first instance, the joint density function factors, and thus the random variables, are independent (with one being exponential with rate 2 and the other exponential with rate 3. In the second instance, because the region in which the joint density is nonzero cannot be addressed in the form $x \in A, y \in B$, the joint density does not factor, so the random variables are note independent.

$$I(x,y) = \begin{cases} 1, if\ 0 < x < 1, 0 < y < 1, 0 < x + y < 1 \\ 0, otherwise \end{cases}$$

And writing $f(x,y) = 24xyI(x,y)$

Independent Random Variables

- We need to define more than two random variables. In general, the n random variables $X_1, X_2, \dots, X_n$ are said to be independent if, for all sets of real numbers $A_1, A_2, \dots, A_n$,

$$P\{X_1 \in A_1, X_2 \in A_2, \dots, X_n \in A_n\} = \prod_{i=1}^{n} P\{X_i \in A_i\}$$

As a before, it can be shown that this condition is equivalent to

$$P\{X_1 \leq a_1, X_2 \leq a_2, \dots, X_n \leq a_n\} = \prod_{i=1}^{n} P\{X_i \leq a_i\}, for\ all\ a_1, a_2, \dots, a_n$$

Independent Random Variables

- **範例九**

令X, Y, Z為獨立與uniform distributed over (0,1). Compute $P\{X \geq YZ\}$.

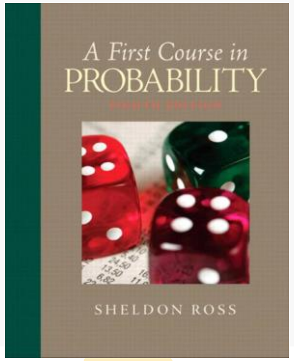
Solution:

Since $f_{X,Y,Z}(x,y,z) = f_Z(x)f_Z(y)f_Z(z) = 1, where\ 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$

$$P\{X \geq YZ\} = \iiint_{x \geq yz} f_{X,Y,Z}(x,y,z)dxdydz$$

$$= \int_0^1 \int_0^1 \int_{yz}^1 dxdydz = \int_0^1 \int_0^1 (1 - yz)dydz = \int_0^1 \left(1 - \frac{z}{2}\right) dz = \frac{3}{4}$$

[#11] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

6.2. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of

(a) X_1, X_2;

(b) X_1, X_2, X_3.

6.3. In Problem 2, suppose that the white balls are numbered, and let Y_i equal 1 if the ith white ball is selected and 0 otherwise. Find the joint probability mass function of

(a) Y_1, Y_2;

(b) Y_1, Y_2, Y_3.

6.10. The joint probability density function of X and Y is given by

$$f(x, y) = e^{-(x+y)} \qquad 0 \leq x < \infty, 0 \leq y < \infty$$

Find (a) $P\{X < Y\}$ and (b) $P\{X < a\}$.

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

Reference

Ross, S. (2010). *A first course in probability*. Pearson.

The End

If you have any questions, please do not hesitate to ask me.

Thank you for your attention))